An analytical model of thermionic discharges

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An analytical model is developed to describe a thermionic plasma discharge. With three easily interpreted algebraic equations, the behavior of the temperature, electron density, and arc drop are determined. Excellent agreement with experiment is found. The model considers a transverse discharge between two electrodes, one of which emits electrons thermionically. Transport of electrons and ions by electric field and density gradient is included. Ions are assumed to be created by volume ionization and are destroyed by recombination either on the walls or by three-body collision in the volume. Conservation of energy for the electrons requires that ohmic heating balance energy losses by collisions and by heat transfer to the walls. Thermionic discharges are also known as low-voltage arcs and occur in some types of thermionic energy converters. The role of the double-valued sheath and Schottky effects is discussed. A possible cause for discharge constriction is mentioned. Also, a “plasma ideal” thermionic discharge is defined and compared with the well-known “vacuum ideal” thermionic diode.

I. INTRODUCTION

This paper describes an electrical discharge in a gaseous plasma between two electrodes, one of which emits electrons thermionically. A distinguishing feature of this type of discharge is its low plasma arc drop. While other types of self-sustained discharges generally have voltage drops larger than an ionization potential, arc drops of 1/2 V or less are common in thermionic discharges. Because of this, these discharges present phenomena of both scientific and engineering interest. The goal of this work is to develop an analytical model which is sufficiently accurate to compare with experiment and yet sufficiently simple to clarify the roles of and relative importance of the various physical phenomena involved.

The engineering interest in thermionic discharges is for power conversion. While other discharges in stationary plasmas consume electricity and produce heat, the thermionic discharge can be made to consume heat and produce electrical power. This can occur when using electrodes which have a difference in work functions larger than the plasma arc drop (see Fig. 1). Such a discharge is called a thermionic energy converter. A temperature difference between the electrodes is essential because its conversion efficiency is limited to less than that of a Carnot engine operating between a thermal reservoir at the cathode temperature and one at the anode temperature. To achieve such a low plasma arc drop, an atomic gas with a very low ionization potential, such as cesium, is used. It is also possible to operate a thermionic energy converter in vacuum without a plasma, but the output power in such operation is limited by space-charge effects.

This theory will consider a plasma with two temperatures, one for electrons and one for heavy particles. Emission of charged particles from the electrodes other than electrons emitted thermionically from the cathode will not be considered. It will be assumed that the length of the electrodes is sufficiently greater than the distance between them so that only a one-dimensional problem need be considered. It will also be assumed that sufficiently many mean free paths separate the electrodes that the plasma may be described by continuum transport equations. Rarefied gas effects, however,

![FIG. 1. The potential energy diagram for a thermionic discharge with output voltage \( V \) is plotted qualitatively against distance in order to define symbols (not to scale). \( \phi_k \) and \( \phi_a \) are the cathode and anode zero-field work functions, respectively. Between the cathode surface at \( x = 0 \) and the quasi-neutral plasma at \( x = 0^+ \) is the cathode sheath. Its thickness, the order of a Debye length, is greatly exaggerated. If, as shown, there is a double-valued sheath at the cathode, the effective work function, \( \psi_{(x)} \), will be larger than \( \phi_k \). If a single-valued sheath is present and the Schottky effect is important, the potential peak at the cathode, \( \phi_k \), will be less than the zero-field value. The anode sheath is located between the anode at \( x = d \) and the quasi-neutral plasma at \( x = d^- \). Its thickness, exaggerated in the diagram, is also of the order of a Debye length. A double-valued sheath does not occur at the anode since its thermionic emission is negligible. The Schottky effect at the anode is also neglected. The cathode and anode sheath voltage drops are \( V_k \) and \( V_a \), respectively. \( \psi(x) \) is the electron potential energy in the plasma. The plasma arc-drop \( V_a \) is defined from sheath peak to sheath peak. \( e \) is the elementary charge.](image-url)
will be included in the formulation of the boundary conditions. These assumptions correspond to the normal operating conditions for an "ignited-mode" thermionic converter.

The difference between the ignited mode and the alternative, variously called "unignited," "extinguished," or "diffusion mode," is in the production of ions. In the ignited mode, ions are dominantly created by ionization in the volume of the plasma. At lower current levels, volume ionization becomes ineffective and the unignited mode occurs for which ionization on an electrode surface dominates. Studies of the unignited mode may be found in Refs. 1 and 4–7 and references therein. Wilkins has studied the transition between the two.8 Thermionic plasmas are further classified according to the gas pressure. The ignited-mode plasma arc drop generally decreases if the gas pressure is high enough that the transport of ions to the walls is impeded by collisions. Such conditions are called "high pressure" even though the pressures involved are typically of order of 0.001 atm. This paper deals with "high-pressure" ignited-mode thermionic discharges.

While there are some similarities between the theory of thermionic discharges and positive columns, there are important differences as well. The presence of thermionic emission is the most obvious one. More important differences are that, in positive columns,9,10 the electric field and current are both perpendicular to the heat flux and ambipolar diffusive flux while, in a thermionic converter, all these quantities are either parallel or antiparallel. Consequently, the equations herein for the arc drop and energy balance do not have analogs in positive column theory.

The motivation for this work was to understand some of the unintuitive and anti-intuitive results of numerical solution of the differential transport equations. This work is an extension of the analytical models of Wilkins and Gyftopoulos,11,12 Lam,13 Hansen, McVey, and Britt,14 and Rasor.5 These analytical models have in common with the present one the approximation that the electron temperature is nearly spatially uniform in the discharge. This is called the "isothermal" approximation even though the electron and heavy particle temperatures are allowed to differ from one another. With some exceptions as will be discussed in Sec. V, this approximation has support from both numerical solutions and experiment. The theory of Lam13 analyzes the plasma arc drop that occurs independently of how the ions are produced. To obtain current-voltage characteristics, Wilkins and Gyftopoulos12 neglect the electric field in the plasma and employ a quasiequilibrium hypothesis under which the electron density is equal to its Saha equilibrium value. Rasor5 also neglects the electric field in the plasma but does not assume quasiequilibrium. In addition, Rasor5 relaxes the isothermal assumption by allowing the electron temperature near the anode to vary. Hansen et al.14 include the electric field but neglect recombination. In the present model, the electric field is included, recombination is included, and no assumption of quasiequilibrium is made.

An example of an unintuitive result first explained by the model herein is the behavior of the electron temperature as the discharge current is changed. It was thought that, since ohmic heating increases as the current increases, the electron temperature should increase also. Numerical computations15,16 sometimes indicated that the electron temperature not only does not increase but may actually decrease significantly as the current is raised. The reasons for this, having to do with the nature of the ionization process, are discussed in Sec. III.

A source of uncertainty in models of thermionic discharges is the behavior of the cathode sheath. This sheath can decrease the effective cathode work function through the Schottky effect. Alternatively, it can increase the effective work function by creating a local potential barrier. Following Ref. 2, a sheath with a local electron potential barrier, as shown in Fig. 1, will be called "double valued" because the same potential can be found at two locations. This has been modeled assuming monoenergetic ions17,18 and assuming shifted partial Maxwellian distributions.19,20 This paper uses the sheath theory of Ref. 20 and includes both the Schottky effect and the double-valued sheath effect but does not consider the possibility of trapped ions.

In Sec. II, the governing differential equations and their boundary conditions are discussed. The basic equations of the model are then developed in Sec. III. An idealized thermionic plasma discharge is developed and compared with the vacuum ideal thermionic diode in Sec. IV. This is followed by comparison of the model with experiment and discussion in Sec. V. Conclusions are drawn in Sec. VI.

II. THE GOVERNING EQUATIONS

The plasma of a thermionic discharge includes three species: electrons, ions, and neutrals. Because the temperatures involved are low, only single-ionized ions need be considered and it may be assumed that the charged-particle densities are much less than the neutral density. The plasma is quasineutral. It will be assumed that the mean-free paths for momentum transfer are much smaller than an interelectrode spacing so that continuum governing equations may be used. Derivation of these equations for the conditions of interest is discussed in detail in Refs. 1 and 21. Diffusion by temperature gradient is neglected.

Transport equations are required for electrons and ions:

\[ \Gamma_e = \frac{\mu_e}{e} \left( -n \frac{d\psi}{dx} kT_e \frac{dn}{dx} \right), \tag{1} \]

\[ \Gamma_i = \frac{\mu_i}{e} \left( n \frac{d\psi}{dx} kT_i \frac{dn}{dx} \right), \tag{2} \]

where \( \Gamma_e \) = electron flux, \( \Gamma_i \) = ion flux, \( e \) = elementary charge, \( n \) = electron density = ion density, \( \psi \) = electron motive, \( k \) = Boltzmann's constant, \( T_e \) = electron temperature, \( T_i \) = ion temperature, \( \mu_e \) = ion mobility, and \( \mu_i \) = electron mobility.

The above transport equations obey the Einstein relation between mobility and diffusivity. The electron motive \( \psi \) is the potential energy of an electron which is often used in place of voltage in electric discharge work because, on a motive diagram, electrons "fall downhill." The voltage is \( -\psi/e \).

To conserve the mass of electrons and ions, some assumption must be made about the ionization/recombination
mechanisms. Early work in thermionic energy conversion determined that direct (single-step) ionization by electron-atom collisions was too slow to explain the observed plasma densities. Consequently, some molecular processes, such as associative ionization, were considered. The work of Bates, Kingston, and McWhirter has established that multistep ionization by electron-atom collisions can be orders of magnitude faster than single-step ionization. Furthermore, multistep electron-atom ionization appears to be faster than any hypothesized molecular ionization process. Thus, the conservation equations for electrons and ions can be written

$$
\frac{d}{dx} \Gamma_e = \frac{d}{dx} \Gamma_i = SN n - \alpha n^3,
$$

where \( S \) is the ionization rate constant, \( \alpha \) is the recombination rate constant, and \( N \) is the number density of ground-state atoms. Various theories give data on \( S \) and \( \alpha \) for conditions of interest in thermionic discharges and these are in agreement with experiment. The behavior of \( S \) and \( \alpha \) depend on radiative escape from the plasma. Thermionic plasmas are typically optically thick to resonance-line radiation but optically thin to line radiation originating from transitions between two excited levels. Because ionization/recombination occurs as a multistep process through the excited levels, radiative decay tends to speed recombination and slow ionization. The relative importance of radiative decay and electron-atom inelastic collisions is determined by the electron density. Consequently, the ionization and recombination rate coefficients are functions of both electron temperature and density

$$
S = S(T_e, n); \quad \alpha = \alpha(T_e, n).
$$

The ratio \( S/\alpha \) will, in general, not be determined by the thermodynamic Saha relation. This is because the kinetics here involve two important unequal temperatures: \( T_e \) and the radiative temperature. If radiative decay is negligible compared to inelastic collisions as might occur if the electron density is high, then the rate coefficients are due to purely collisional kinetics and may be approximated as being functions of just electron temperature:

$$
S = S(T_e); \quad \alpha = \alpha(T_e).
$$

In this limit \( T_e \) is the only important temperature and the Saha relation applies. General discussions on the nature of collisional-radiative electron-atom ionization/recombination can be found in Refs. 27, 28. In addition to thermionic discharges, multistep ionization/recombination has been found to be important in situations ranging from astrophysical plasmas, to low-pressure positive columns, to high-pressure arcs.

The ionization coefficient \( S \) is sensitive to the electron energy distribution. Because of the large cross sections for vibrational-rotational inelastic collisions, the electron energy distribution is often strongly non-Maxwellian in molecular plasmas. In an atomic plasma, such processes are absent and electron-electron collisions are often sufficient to cause a Maxwellian distribution. Most thermionic discharge experiments have been performed in alkali-metal gases at sufficiently high temperatures that they are atomic. Studies of the electron energy distribution under these conditions show that it is very close to Maxwellian except possibly at the lowest electron densities considered. The calculations in this paper assume a Maxwellian distribution.

The above mass and transport equations for ions and electrons are more easily solved if rearranged as follows. Substituting Eq. (2) into Eq. (3) yields a conservation equation for ions

$$
\frac{d}{dx} \left( \frac{\mu_i}{\epsilon} \left( -kT_i \frac{dn}{dx} + n \frac{d\psi}{dx} \right) \right) = SN n - \alpha n^3.
$$

Solving Eq. (1) for the electric field term gives

$$
n \frac{d\psi}{dx} = -kT_e \frac{dn}{dx} - e\Gamma_e / \mu_e.
$$

Using electron transport Eq. (7) to eliminate the electric field term from the ion conservation Eq. (6) yields a diffusion equation:

$$
\frac{d}{dx} \left( -D_e \frac{dn}{dx} \right) - \frac{d}{dx} \left( \frac{\mu_i}{\mu_e} \right) = \left(1 + \frac{\mu_i}{\mu_e} \right) \left( SN n - \alpha n^3 \right),
$$

where

$$
D_e = \mu_i (kT_e + kT_i) / e.
$$

Assuming

$$\mu_i / \mu_e \ll 1$$

and if variations in the ratio \( \mu_i / \mu_e \) can be neglected, then the diffusion equation simplifies to

$$
0 = \frac{d}{dx} \left( D_e \frac{dn}{dx} \right) + SN n - \alpha n^3.
$$

The above equation will play a major role in the model to be developed in Sec. III. This equation has the form of a simple diffusion equation with a source term. It indicates that the plasma behaves as if it is created by ionization, destroyed by recombination, and it diffuses with flux \( -D_e \frac{dn}{dx} \). It should be remembered however that, in general, \( -D_e \frac{dn}{dx} \) does not represent the flux of any physical quantity, as can be seen by comparison with Eq. (1) and Eq. (2). Some authors call an equation such as Eq. (11) an ambipolar diffusion equation, others reserve the term ambipolar diffusion for conditions with no current. As follows from the above derivation, Eq. (11) remains valid both with and without net current flow.

In Eq. (2), electron drag was neglected. If it was included however, Eq. (11) would remain unchanged. This is because electron drag in the ion transport equation would cancel with the corresponding ion drag term in the electron transport equation after the substitution used to eliminate the electric field.

The boundary conditions for this plasma are found assuming \( \lambda / d < 1 \) where \( \lambda \) is a momentum transfer mean free path and \( d \) is the interelectrode distance. The boundary conditions on the plasma transport equations are applied in the quasineutral plasma near the interface with the electrostatic sheath. Since the sheaths in this type of discharge are very thin, the order of a Debye length, the \( x \) positions at which the boundary conditions are applied are denoted \( 0^+ \) and \( d^- \). In the lowest-order approximation, the walls can be consid-
ered absorbing and the boundary conditions on electron density in the simplest approximation are
\[ n(0 + ) = 0, \]
\[ n(d - ) = 0. \]

Boundary conditions are discussed in more detail in the Appendix.

Another consequence of mass conservation, Eq. (3), is current conservation:
\[ \frac{d}{dx} (\Gamma_e - \Gamma_i) = 0. \]

Consistent with the assumption of small ion mobility, Eq. (10), the electrons carry most of the current and the above equation simplifies to
\[ \frac{d}{dx} \Gamma_e = 0. \]

An energy equation is needed for electrons. In steady state:
\[ \Gamma_e \frac{d}{dx} \left( \frac{5}{2} kT_e + \psi \right) - \frac{d}{dx} \left( \frac{K}{d} \frac{d}{dx} T_e \right) = - S^{(E)}, \]
where \( K \) is the thermal conductivity and \( S^{(E)} \) is a collisional energy-sink term which may include elastic, inelastic, and supereelastic collisional energy transfer with ions and neutrals. The boundary conditions for the electron energy equation are found by equating the energy flux in each sheath with the energy flux in the adjacent plasma. At \( x = 0 + \):
\[ \Gamma_e \left( \frac{5}{2} kT_e + \psi \right) - K \frac{d}{dx} T_e = \Gamma_i \left( \frac{5}{2} kT_i + \psi \right) - (\Gamma_e - \Gamma_i) \left( \frac{5}{2} kT_e + \psi \right) \]
\[ - \frac{d}{dx} \left( \frac{K}{d} \frac{d}{dx} T_e \right) \]
and, for \( x = d - \):
\[ \Gamma_e \left( \frac{5}{2} kT_e + \psi \right) - K \frac{d}{dx} T_e = \Gamma_i \left( \frac{5}{2} kT_i + \psi \right) - \frac{d}{dx} \left( \frac{K}{d} \frac{d}{dx} T_e \right), \]

where \( T_e \) is the temperature of the emitted electrons which equals the cathode temperature. In these equations, the terms on the left are the energy fluxes in the plasma near each sheath. These consist of convection and conduction terms. The terms on the right are the energy fluxes through the sheaths. The flux of electrons thermionically emitted by the cathode is \( \Gamma_e \). The energy carried by these electrons into the plasma is represented by the first term on the right in Eq. (15). Since the net electron flux is less, a flux of electrons, \( \Gamma_i - \Gamma_e \), must leave the plasma and return to the cathode. The energy carried by these electrons is represented by the second term on the right in Eq. (15). Since no anode emission is assumed, there is only one term on the right in Eq. (16) and it represents the energy carried by the flux \( \Gamma_e \) of electrons leaving the plasma and entering the anode.

In summary, the governing equations for a thermionic discharge have been stated. These are transport, Eq. (1) and Eq. (2), conservation of mass, Eq. (3), and conservation of electron energy, Eq. (14). Because the diffusion Eq. (11) separates the electrical and transport problems, the actual set of equations to be solved will be the diffusion balance, Eq. (11), the electron transport relation, Eq. (1), the electron energy balance, Eq. (14), and current conservation, Eq. (13). The diffusion equation is subject to boundary conditions Eq. (12) and the electron energy equation to boundary conditions Eq. (15) and Eq. (16). Except for the neglect of temperature gradient, these equations include effects of varying temperature. These equations are valid under the assumptions of (i) small mean free path, \( \lambda / d \ll 1 \), (ii) steady operation, (iii) quasineutral plasma, (iv) small ionization fraction, \( n/N \ll 1 \), and (v) the ratio of ion to electron mobility is small and approximately constant. These assumptions are accurate for the normal operating conditions of an ignited-mode thermionic energy converter.

III. THE MODEL

This model will show that the main features of thermonic discharges can be described by three algebraic relations. The first is a condition for ignited-mode operation which determines the electron temperature necessary to provide enough ionization to maintain the plasma. The second uses global energy conservation to determine the plasma arc drop. The third determines electron temperature from balancing the heating and cooling mechanisms. These three relations put together allow the current-voltage characteristics, electron temperatures, and electron densities of thermonic discharges to be determined. The main additional assumption used to develop this model is that the electron temperature is approximately uniform. As will be discussed in Sec. V, this assumption, with some exceptions, is in reasonable agreement with numerical theory and experiment.

A. The ignited condition

The charged-particle diffusion Eq. (11) expresses a balance of ionization with diffusion to the walls and recombination. This equation will be used to find the electron temperature and the density distribution. These results will be found by analyzing the diffusion equation as a (nonlinear) eigenvalue problem.

While more general cases will be considered later in this section, a simple case of diffusion is discussed first in which the cesium density is uniform, the diffusion coefficient is constant, recombination negligible, and the ionization kinetics purely collisional. Within the uncertainty of cesium momentum-transfer cross sections, the assumption of a constant diffusion coefficient is quite reasonable. Recombination can be neglected when the electron density is much less than its Saha value. This occurs in practice when the pd (pressure-distance) product is not too large or the current is not too high. The assumption of purely collisional ionization/recombination kinetics again means the use of Eq. (5) rather than Eq. (4). Hence, under the isothermal approximation, \( S \) and \( \alpha \) may be considered independent of position. Under these assumptions, the ambipolar diffusion equation is linear and has constant coefficients:
\[ O = D_n \frac{d^2 n}{dx^2} + SN n. \]

\[ \text{1878 J. Appl. Phys., Vol. 59, No. 6, 15 March 1986} \]
With the boundary conditions of Eq. (12), this is an eigenvalue problem. For a nonzero solution, it is required that

\[ S(T_e)N = D_e \frac{n^2}{d^2}, \]  

(18)

where \( d \) is the interelectrode distance. Since \( S \) is an extremely sensitive function of electron temperature \( T_e \), the above relation may be thought of as determining \( T_e \). If Eq. (18) is satisfied, the electron density is

\[ n(x) = n_{\text{MAX}} \sin(\pi/d)x \quad \text{for} \quad 0 < x < d. \]  

(19)

If Eq. (18) is not satisfied then Eq. (17) does not have a nontrivial steady solution. If the left-hand side of Eq. (18) exceeded the right, ionization would occur faster than diffusion to the walls and the ion density \( n \) would increase. If the left-hand side were smaller than the right, ionization would be insufficient to balance diffusion and the density would decay. An equation such as Eq. (19) is a condition for steady ignited-mode operation and, thus, will be called an ignited condition. Later sections will indicate how \( T_e \) changes to satisfy that the eigenvalue condition is satisfied.

Observe that the ambipolar diffusion Eq. (17) has determined the electron temperature through Eq. (18) and the shape of the electron density distribution, Eq. (19), but leaves the magnitude of the electron density, described by the multiplicative constant \( n_{\text{MAX}} \), undetermined. This quantity will be determined later through energy balance considerations.

Analogous situations in which a critical ionization rate constant is necessary for ionization to balance diffusion occur frequently. Examples include positive columns,37 microwave discharges,37,38 and thin channel MHD generators.35

The effect of recombination on the above results can be assessed as shown in Fig. 2. The Saha equilibrium relation between electron temperature and electron density is plotted as curve 3. On this line, the electron temperature is just high enough that the ionization rate balances recombination. Sufficiently above or to the left of this line, recombination may be neglected. The ignited condition, Eq. (18), labeled as curve 1 on Fig. 2, shows that recombination is negligible over a large range of electron densities but must be included at higher densities which, as will be seen later, correspond to higher currents. A more accurate ignited condition which includes collisional-radiative ionization and recombination will be developed later in this section.

The dependence of the electron temperature on other variables can be examined if a simple form is used to describe the ionization coefficient. This simple form is presented for illustrative purposes only and more accurate values will be used in later quantitative calculations. Over a limited range of temperature

\[ S(T_e) \approx A_0 \exp(-E_a/kT_e), \]  

(20)

where \( A_0 \) and \( E_a \) are constants. Combining Eq. (20) with the eigenvalue condition, Eq. (18), yields

\[ T_e = \left( \frac{E_a}{k} \right) \ln \left( \frac{A_0 N d^2}{\pi^2 D_e} \right). \]  

(21)

\( T_e \) is thus determined by the ability of the discharge to create and retain ions. Thus, as the diffusivity \( D_e \) increases or the distance \( d \) that the ions must diffuse decreases, \( T_e \) must increase to provide faster ionization. The lower the neutral density \( N \), the fewer the electron-atom collisions and consequently the hotter the electrons must be to provide the same rate of ionization. Since \( D_e \propto 1/N \), the argument of the logarithm scales with \( N d \) which explains, in part, why the pressure-distance product can be used to correlate data. For a cesium thermionic-energy converter, typical electron temperatures are in the range of 2500–3000 K, roughly twice the typical electrode temperature.

The above solution was found by assuming constant coefficients in the diffusion equation. A more precise ignited condition could be found by considering small variations of \( D_e \) and \( N \) with position. This could be done using independent variable transformations and the WKB method. Since Eq. (21) indicates that \( T_e \) has only a weak logarithmic dependence on these quantities, however, such corrections are not likely to be important.

Recombination can be important in the diffusion balance. If it is included the equation is no longer linear:

\[ O = D_e \frac{d^2n}{dx^2} + SNn - an^3. \]  

(22)

The solution to this equation is given by Jacobian elliptic functions.39 In this case, the electron density is

\[ n(x) = n_{\text{MAX}} \sin[(2K/d)x], \]  

(23)

where \( K \) is the quarter period of the Jacobian elliptic function \( n_{\text{MAX}} \) retains its meaning as the maximum ion density in the discharge. The previous eigenvalue condition, Eq. (18), is replaced by

\[ S(T_e)N = (1 - m)(2K/d)^2(D_e/d^2) + \alpha(T_e) n_{\text{MAX}}^2. \]  

(24)

This shows what ionization rate is necessary to balance diffusion and recombination. The main qualitative difference
caused by including recombination is that the electron temperature $T_e$, determined by Eq. (24), as opposed to Eq. (18), depends on $n_{\text{MAX}}$. The quarter period $K$ is a function of the parameter $m$ of the Jacobian an elliptic function. For this case:

$$m = \frac{1}{(2SN)/(\tan^{2}n_{\text{MAX}}) - 1}. \quad (25)$$

The legal range of $m$ is $0 < m < 1$. Calculation of $K$ as a function of $m$ is very easily and accurately performed using the arithmetic-geometric mean method described in Sec. 17.6 of Ref. 39. Calculation of $n_{\text{SN}}$ is similarly performed as described in Sec. 16.4 of Ref. 39.

In the limit of small $m$, the previous solution, Eq. (18) and Eq. (19), is recovered. In the limit of $m \to 0$, then $K \to \pi/2$ and the function $s_n$ approaches $\sin$. Recombination is important in the opposite limit of $m \to 1$. In this case $(1 - m) (2K)^2 \to 0$ indicating that diffusion is unimportant and the ignited condition, Eq. (24), becomes simply a balance between ionization and recombination.

An additional new feature is found by including collisional-radiative kinetics in Eq. (4) as opposed to the purely collisional kinetics in Eq. (5). Such a functional form for the coefficients could make the diffusion equation much more difficult to solve. $S$ and $\alpha$, fortunately, are slowly varying functions of $n$. Furthermore, since most of the ionization and recombination occurs at large electron densities, a reasonable approximation is obtained by simply evaluating the coefficients $S$ and $\alpha$ for an electron density of $n_{\text{MAX}}$. Thus Eq. (24) is replaced by

$$S(T_e, n_{\text{MAX}}) = (1 - m) (2K)^2 (D_e/d^2) + \alpha(T_e, n_{\text{MAX}}) n_{\text{MAX}}. \quad (26)$$

This is the most accurate form of the ignited condition that will be presented in this paper and is the form that will be used in later quantitative calculations.

The importance of collisional-radiative ionization and recombination is shown in Fig. 2. If both are neglected, the ignited condition is given by Eq. (18) and is independent of electron density. It is labeled as curve 1 in Fig. 2. If both are included, Eq. (26) applies and the plot of this is labeled curve 2 in Fig. 2. At high electron densities, $T_e$ must rise for ionization to balance recombination, as shown in Fig. 2. At the highest densities shown, loss by diffusion is negligible and the temperature determined by Eq. (26) is asymptotic with the Saha relationship between electron temperature and density, shown as curve 3. For electron densities of about $10^{14}$ cm$^{-3}$, diffusion to the walls is important and Eq. (26) agrees closely with Eq. (18). At still lower electron densities, these latter two curves diverge. This is because of collisional-radiative ionization kinetics. At lower electron densities, radiative decay inhibits ionization and a higher temperature is needed for ionization to balance diffusion. Combined with later results, this will explain why electron temperatures increase at low currents.

The results of this section can be compared with approximation used in previous models. Wilkins and Gytopoulos approximated the ignited condition with the Saha equation. Figure 2 shows this to be accurate at high-electron densities.

Later, Wilkins used an equation similar to Eq. (24). Rassor's model assumed $T_e$ to be the higher of that given by Eq. (18) or by the Saha relation. None of these previous models considered collisional-radiative kinetics.

The density distributions above differ qualitatively from previous models which neglected the plasma electric field. The distribution of Eq. (19) or Eq. (23) peaks at the center, $x = d/2$, and decays toward each boundary. By contrast, the density distribution of Ref. 5 and Ref. 12 are monotonically decreasing from cathode to anode. (See Eqs. (12) and (13) of Ref. 5 and Eqs. (16) and (16a) of Ref. 12. This monotonous decrease results from neglecting $d\psi/dx$ in the electron transport equation, Eq. (1) of this paper. The distribution herein results from Eq. (11) which implicitly includes the electric field, as is apparent from its derivation.

The model assumes a spatially uniform electron temperature. The results, fortunately, are not sensitive to this assumption. With a varying electron temperature, the ionization coefficient $S$, would vary with position. This effect has been investigated. A generalized ignited condition can be found which applies to a weighted average of $S$ over the discharge. The results are quantitatively quite similar.

In summary, it has been shown that there is an electron temperature necessary for ionization to balance recombination and diffusion to the walls. This condition is called the ignited condition because it must be satisfied for the discharge to operate in the "ignited mode," that is, with a plasma sustained by volume ionization. This condition was given in various approximations by Eqs. (18), (21), (24), and, most accurately, by Eq. (26).

**B. Global energy balance**

The global conservation of energy equation to be developed herein is very useful for predicting the plasma arc drop. This equation will first be developed in a general nonisothermal form so that the effect of making the isothermal approximation can be clearly understood. Integrating the conservation of energy Eq. (14), over the plasma, and using boundary conditions Eqs. (15) and (16), yields

$$\left[\psi(0^+) + eV_K + 2kT_e(0^+)\right] (\Gamma_E - \Gamma_e) - \left[\psi(0^+) + eV_K + 2kT_e\right]\Gamma_E$$

$$+ \left[\psi(d^-) + eV_A + 2kT_e(d^-)\right]\Gamma_e$$

$$+ \int_0^\infty S_{(E)}\,dx = 0. \quad (27)$$

This equation can be simplified as follows. From Fig. 1, the quantity $\left[\psi(0^+) + eV_K\right]$ is seen to be equal to the effective cathode work function $\phi_K$. $\left[\psi(d^-) + eV_A\right]$ is equal to $(\phi_e - eV_d)$ where $V_d$ is the voltage drop across the plasma.

The global energy balance can be rearranged and solved for the plasma arc-drop $V_d$:

$$V_d = 2(\Gamma_E/\Gamma_e) \left(k/e\right) \left[T_e(0^+) - T_K\right] + Q/e\Gamma_e$$

$$+ 2(k/e) \left[T_e(d^-) - T_e(0^+)\right], \quad (28)$$

where
Further invoking the isothermal approximation,

\[ T_e = T_e (0 + ) \approx T_e (d - ) . \]

The energy Eq. (28) reduces to

\[ V_d = 2 (J_E / J) (k / e) (T_e - T_K) + Q / J , \]

where \( J_E = e \Gamma_e \) and \( J = e \Gamma_e \). Equation (28) gives the arc drop for a nonisothermal plasma. Comparison of Eq. (28) and Eq. (30) shows the effect of making the isothermal assumption on the arc drop.

Equation (30) has very important implications for the arc drop and is one of the three basic equations of the analytical model. Physically, it says that the voltage drop across the plasma \( V_d \) must generate enough heat \( J V_d \) to balance power losses by heat transfer to the walls, \( 2 (k / e) J_E (T_e - T_K) \), and by collisional losses \( Q \). Thus \( V_d \) depends on three quantities: (i) \( J_E / J \), (ii) \( T_e - T_K \), and (iii) \( Q / J \). Schemes to reduce the arc drop may be evaluated by how they affect these three.

### C. Temperature equation

So far, the model consists of two equations, an ignited condition, Eq. (26), and an energy balance, Eq. (30). The expression for the plasma arc drop [Eq. (30)] was developed without reference to the dissipation mechanisms in the plasma. To close the model, such mechanisms must be examined. To do this, the electron transport Eq. (1) and sheath drops will be considered. These will produce another expression for the plasma arc drop based on ohmic losses, sheath drops, and a diffusive loss. This expression will be developed using the isothermal assumption. Combined with the energy balance Eq. (30), the dependence of electron temperature on ohmic heating, inelastic collisions, and sheaths is determined.

The electron transport Eq. (1) can be solved for the electric field:

\[ \frac{d \psi}{dx} = - \left( \frac{e \Gamma_e}{n \mu_e} + \frac{k T_e}{n} \right) . \]

Under the isothermal assumption, this can be integrated from the edge of the plasma near the cathode sheath \( x = 0 + \), to the plasma edge near the anode sheath \( x = d - \), yielding

\[ \psi (d - ) - \psi (0 + ) = - \int_{0 + }^{d - } \frac{\Gamma_e}{n \mu_e} dx - \frac{k T_e}{e} \ln \frac{n (d - )}{n (0 + )} . \]

The first term on the right-hand side represents ohmic losses. The second represents the voltage change necessary to overcome diffusion by density gradient and is called the diffusive arc drop.

The sheath drops \( V_K \) and \( V_d \) can be found as described in the Appendix:

\[ V_K = \frac{k T_e}{e} \ln \frac{n (0 + ) \bar{v}_e / 4}{\Gamma_e} , \]

\[ V_d = \frac{J E}{J} \ln \frac{J}{J E} . \]

Combining the expressions for the plasma arc drop, Eq. (31) with the sheath drops Eq. (32), the total arc drop can be found:

\[ V_d = \int_{0 + }^{d - } \frac{\Gamma_e}{n \mu_e} dx + \frac{k T_e}{e} \ln \frac{\Gamma_e}{\Gamma_e - J} . \]

Equation (33) can be conveniently rewritten:

\[ V_d = J R + \frac{k T_e}{e} \ln \left( \frac{J}{J E - J} \right) . \]

where \( J = e \Gamma_e \), \( J_E = e \Gamma_e \), and \( R \) is the resistance of the plasma over a unit area of electrode surface:

\[ R = \int_{0 + }^{d - } \frac{dx}{n \mu_e} . \]

Evaluation of this integral will be performed later. The first term on the right-hand side in Eq. (34) is the ohmic loss. The second results from a combination of the cathode sheath voltage, the anode sheath voltage, and the diffusive arc drop. Observe that, in combining the sheath and diffusive voltage drops, several terms canceled. In particular, the combined term no longer depends on the plasma densities at \( x = 0 + \) and \( x = d - \).

Equation (34) gives an accounting for the arc drop independent of that found from the energy balance Eq. (30). Since Eqs. (34) and (30) are both valid expressions for \( V_d \), they may be equated:

\[ \frac{2 J E}{J} \left( \frac{k}{e} \right) (T_e - T_K) + Q / J = J R + \frac{k T_e}{e} \ln \left( \frac{J}{J E - J} \right) . \]

This may be rearranged to give a useful equation for \( T_e \):

\[ T_e = T_K + \left( \frac{e}{k} \right) (J^2 R - Q) / (2 J_E) , \]

\[ 1 + 1 / 2 \left( \frac{J}{J_E} \right) \ln \left( \frac{J}{J E - J} \right) . \]

This is an important relation because it shows phenomenologically how plasma resistance, electron-collisional energy loss, and the cathode sheath behavior affect the electron temperature. This determination of the electron temperature is based on electron momentum, energy, and sheath considerations and is independent of the ignited condition which is based on conservation of mass. The denominator of Eq. (37) is positive when \( J / J_E < 0.902 \). This signifies that the coefficient of \( T_e \) in the wall cooling term on the left of Eq. (36) is larger than the coefficient of \( T_e \) in the sheath/diffusive term on the right of Eq. (36). If \( J / J_E < 0.902 \), Eq. (37) shows that ohmic heating, \( J^2 R \), tends to increase the temperature while the heat sink \( Q \) tends to decrease it, both of which are physically intuitive conclusions. The cathode sheath affects the electron temperature because it controls the emitted current \( J_E \) which appears in Eq. (37).

With Eq. (37), the discharge operating conditions can be found. For a given current \( J \), Eq. (37) may be used to plot \( T_e \) against \( n_{MAX} \). The ignited condition may be placed on the same plot, as shown in Fig. 3. Since both are simultaneously valid, an intersection of the two curves represents a possible operating point for the steady thermionic discharge.
The behavior of Eq. (37) as shown in Fig. 3 can be explained as follows. First, consider a fixed current density, \( J \). As \( n_{\text{MAX}} \) increases, three quantities change: \( R \) decreases, \( Q \) increases, and \( J_E \) increases. The reasons \( R \) and \( Q \) change are discussed in the following subsections. The change in \( J_E \) is due to the sheath and occurs either in the double-valued sheath mode or in the Schottky enhanced mode. When \( J/J_E < 0.902 \), these three changes all tend to decrease \( T_s \). Consequently, on a plot of \( T_s \) vs \( n_{\text{MAX}} \), Eq. (37) typically has a large negative slope as in Fig. 3.

D. Plasma resistance \( R \)

The electrical resistance \( R \) is important in the temperature equation, Eq. (37), and will be evaluated in this subsection. The form of the electrical resistance \( R \), defined by Eq. (35), depends on the electron scattering mechanisms. For large ionization fractions, Coulomb scattering of electrons off ions will dominate and \( R \) has the simple form

\[
R_{el} = \frac{d}{\sigma(T_s)},
\]

(38)

where \( d \) is the gap length and where \( \sigma(T_s) \) is the electrical conductivity as given by Spitzer and Harm.\(^{42,43}\)

If electron-neutral scattering dominates, the calculation of \( R \) is slightly more subtle. With a density distribution such as Eqs. (19) or (23), satisfying boundary conditions Eq. (12), the integral in Eq. (35) is singular. This is because a zero-electron density at the boundary implies an infinite resistivity. The infinite resistivity does not exist because the boundary conditions Eq. (12) are only approximations. To determine the value of \( R \), the diffusion Eq. (22) must be solved subject to the more accurate boundary conditions Eq. (50) described in the Appendix. This results in

\[
R_{en} = \frac{d}{2K\eta_{\text{MAX}} \mu_e} \ln \left( \frac{4(n_{\text{MAX}})^2}{(1-m)n(0+)+n(d-)} \right),
\]

where \( K \) is the quarter period of an elliptic function with parameter \( m \) as defined by Eq. (25). To simplify the above result, it was assumed that \( \eta \) the electron mobility is a constant, and (2) \( n(0+)/n_{\text{MAX}} \) and \( n(d-)/n_{\text{MAX}} \) are both much smaller than one, as is implied by Eq. (50) and the assumption of a small mean-free path. Eliminating \( n(0+) \) and \( n(d-) \) using Eq. (50) yields

\[
R_{en} = \frac{d}{2K\eta_{\text{MAX}} \mu_e} \ln \left( \frac{4n_{\text{MAX}}^2}{(1-m)n(0+)+n(d-)} \right).
\]

(39)

For intermediate cases, both electron-ion and electron-neutral may be important. To a first approximation, the expressions for resistance in the two limits may simply be added:

\[
R \approx R_{el} + R_{en}.
\]

(40)

This is because of superposition rules for mobilities and is often accurate to 10\%.\(^1\) In the special case for which the electron-neutral cross section varies rapidly with impact energy over the thermal range, however, the simple superposition rule may be inaccurate.\(^{44}\)

\( R \) plays an important role in the temperature Eq. (37). For the weakly ionized case, Eq. (39) shows that \( R \) decreases with increasing \( n_{\text{MAX}} \).

E. Plasma heat-sink \( Q \)

The temperature equation, Eq. (37), requires a value for the plasma heat-sink \( Q \). \( Q \) is defined by Eq. (29) and will be evaluated in this subsection. Electrons in the bulk of the plasma lose energy by collisions with atoms. This energy results in (1) heating the translational modes of the heavy particles, (2) radiative escape, and (3) ionization. Considering first just the third, the heat-sink \( S^{(e)} \) can be written

\[
S^{(e)} = E_+ (S N n - a n^3),
\]

where \( E_+ \) is the ionization energy. To find \( Q \), as defined by Eq. (29), \( S^{(e)} \) must be integrated over the plasma. From diffusion Eq. (11), it follows that

\[
Q = E_+ \left( \frac{d}{dx} \frac{dn}{dx} \right)_{x=0+} - E_+ \left( \frac{d}{dx} \frac{dn}{dx} \right)_{x=d-}.
\]

The density gradient may be found from an equation Eq. (23) and then

\[
Q = 2E_+ (2KD_e/d) n_{\text{MAX}}.
\]

Thus, the energy-loss \( Q \) is an increasing function of \( n_{\text{MAX}} \).

Escape of resonance radiation and cooling by elastic collision with heavy particles can be easily included in the heat sink:

\[
Q = 2 \left[ E_+ A_K (2KD_e/d) \right] n_{\text{MAX}} + g A_{2-1} N_2 d
\]

\[\quad + (3m_e/M) \bar{n} \nu_{\text{eff}} (kT_e - kT_H) d.
\]

(41)

Here, \( A_{2-1} \) is the Einstein \( A \) coefficient for the rate of spontaneous emission of photons of energy \( E_{2-1} \) from the resonance level, of average number density \( N_2 \), to ground. \( g \) is the escape factor for such radiation,\(^{45-47}\) and \( d \) is the interelectrode distance. The energy transferred to an elastic collision is proportional to the \( m_e/M \), the ratio of the electron mass to the mass of a heavy particle, and to the temperature difference \( T_e - T_H \) between electrons and heavy particles. The collision frequency between electrons and heavy particles,
\( \nu_{th} \) is proportional to the heavy particle density. \( \bar{n} \) is the average electron density in the plasma and is related to \( n_{\text{MAX}} \) for the distribution in Eq. (23) by

\[
\bar{n} = \frac{1}{m^{1/2} K} \ln \left( \frac{1 - m^{1/2}}{1 - m^{1/2}} \right) n_{\text{MAX}}.
\]

For the calculations that follow, energy losses to ionization and radiation are included while energy losses by elastic collisions with heavy particles are neglected as small under the conditions of interest.

F. Summary of the model equations

In its final form, the model consists of three algebraic equations describing basic physical phenomena. The first is the ignited condition. This determines what temperature is necessary for ionization to balance recombination and diffusion to the walls. This condition appeared mathematically as an eigenvalue condition on the diffusion equation. The general form of the ignited condition is Eq. (26). Various special cases of this were discussed including Eqs. (24), (18), and (21).

The second important equation is Eq. (37) which determines the electron temperature for which the heating and cooling mechanisms are in balance. Combined with the ignited condition this allowed the electron density and temperature to be determined as a function of current, as shown in Fig. 3. To evaluate the temperature equation numerically, it is necessary to have expressions for the resistance \( R \) as in Eq. (40) and for the heat-sink \( Q \) as in Eq. (41).

The third important equation is Eq. (30) which determines the arc drop \( V_d \) necessary for energy in to the plasma electrons to balance energy losses to the walls and to collisions with heavy particles. The losses to the walls occur because the plasma electrons at temperature \( T_e \) are exchanged with cooler electrons from the cathode at temperature \( T_K \) at a rate determined by the emitted current \( J_e \).

The quantitative results of this theory will be presented in Sec. V.

IV. IDEALIZED DISCHARGE

The equations in the previous section can give a quantitative description of the current-voltage characteristics of a thermionic discharge. The purpose of this section is to explore a limiting case of discharge behavior under "ideal" conditions. While such conditions are not physical, they will clarify the essential loss mechanisms of a plasma discharge and permit comparison with the well-known "vacuum ideal" thermionic diode.

The "plasma ideal" discharge will be defined as a discharge with a lossless and independently sustained plasma. In particular, this means that (1) the plasma will have no electrical resistance, \( R = 0 \), (2) the plasma has no collisional energy sink, \( Q = 0 \), and (3) ions neither recombine nor diffuse to the walls so that the plasma continues to exist even if the ignited condition is not satisfied. Under these assumptions, the temperature equation, Eq. (37), reduces to

\[
T_e = \frac{T_K}{1 + 1/2 (J/J_K) \ln(J/K - 1)}.
\]

For 0.5 \( J/J_K < 0.902 \), the above equation predicts \( T_e \) to be higher than the cathode temperature \( T_K \). This is because the cathode sheath is large and the anode sheath small for these conditions and the current flow results in a net heating. In the limiting case of \( J/J_K \rightarrow 0.902 \), this heating becomes intense and Eq. (42) predicts \( T_e \rightarrow \infty \). For 0 < \( J/J_K < 0.5 \), the situation is reversed, the anode sheath becomes large, and the electrons are cooled to a temperature below \( T_K \).

With the above expression for \( T_e \), the plasma arc drop \( V_d \) can be determined. Using Eq. (42) and either Eq. (30) with \( Q = 0 \) or Eq. (34) with \( R = 0 \), results in

\[
V_d = \frac{kT_K}{e^{1/2 (J/K - 1)}} \ln(J/K - 1).
\]

This is valid only for the "plasma ideal" discharge. Equation (43) indicates that the nondimensional arc drop \( eV_d/kT_K \) for the plasma ideal discharge is a universal function of \( J/J_K \). This arc drop is due to the differences in the sheath voltages and the diffusion arc drop.

It is instructive to compare the plasma ideal current-voltage characteristics with the vacuum ideal behavior. A vacuum ideal thermionic diode has an interelectrode spacing sufficiently small that the space-charge effects are negligible.1,2,3 It is customary to compare the plasma discharge behavior with that of a vacuum diode with the same zero-field cathode work function, \( \phi_K \). Here, it is more convenient to compare with a vacuum diode with the same effective work function \( \phi_K \) as the plasma discharge. In this case, the vacuum ideal current-voltage relation is

\[
J = \begin{cases} \frac{J_K}{e^{V_d/kT_e}} & \text{if } V_d > 0 \\ \frac{J_K}{e^{V_d/kT_e}} & \text{if } V_d < 0 \end{cases}
\]

Equation (44) indicates that \( eV_d/kT_K \) is also a universal function of \( J/J_K \) for the vacuum ideal diode.

The vacuum ideal and plasma ideal curves are compared in Fig. 4. It is seen that the plasma ideal discharge is inherently more lossy than the vacuum ideal diode. This is primarily due to the role of the plasma in backscattering emitted electrons. This necessitates a cathode sheath drop which reduces the overall output voltage. This loss exists even in the "plasma ideal" discharge which has no resistance and no ionization losses.

V. RESULTS AND DISCUSSION

In this section, the results of the isothermal theory of Sec. III will be compared with experiment and with other theories. The relative importance of various contributions to the arc drop will be analyzed. Also, the importance of the Schottky and double-valued sheath effects will be assessed.

The major assumption of the present theory is that the electron temperature is approximately uniform across the plasma. To assess the validity of this assumption, comparison will be made with a numerical solution of the nonisothermal differential equations, Eqs. (1), (8), and (14).16 (This is shown in Figs. 5 and 6). The numerical calculations show that \( T_e \) is higher at the cathode than at the anode. The change however is slight and the isothermal prediction is very close. Similarly, the density distributions predicted by both theories are quite close. Figure 6 shows that the non-
isothermal calculation predicts a slightly higher density with the peak shifted slightly towards the cathode. Agreement between isothermal and numerical calculations, however, is not always as close as shown in Figs. 5 and 6. Numerical calculations sometimes predict a temperature change of as much as \( \sim 1500 \text{ K} \) across the discharge.\(^2\)

Direct experimental measurements of the temperature profile have a large amount of scatter because of the difficult diagnostics involved.\(^{1,33,48-50} \) Like the numerical calculation, they indicate that the electron temperature declines from cathode to anode. The decline is typically less than \( \sim 400 \text{ K} \). The experimental uncertainties are \( \sim 200 \text{ K} \).

Comparison with experimental current-voltage curves provide another test of the isothermal assumption. Figure 7 compares experiments performed by Rufe and Lieb\(^5\) with the present theory. This set of experimental data was chosen as the most accurate set of thermionic discharge data by McVey and Brit.\(^5\) The theory was calculated using elastic cross sections of \( 4.0 \times 10^{-14} \text{ cm}^2 \) for electron-atom collisions and \( 1.2 \times 10^{-13} \text{ cm}^2 \) for ion-atom collisions.\(^1,52\) The experiment was performed using a duplex vapor-deposited tungsten cathode at 1700 K and a molybdenum anode at 900 K with an interelectrode spacing \( d \) of 0.0254 cm (10 mil). For cesium pressures of 130, 270, and 400 Pa (1, 2, and 4 Torr), the calculations used anode work functions of 1.61, 1.67, and 1.77 eV, respectively. These are within the experimental scatter on the Rasor plot of work functions measured by retarding plot and back-emission methods.\(^5\) Space-charge limitations prevented direct measurement of cathode work functions under the conditions of interest. The values chosen for calculation were 2.76, 2.60, and 2.47 eV for Cs pressures of 130, 270, and 400 Pa (1, 2, and 4 Torr), respectively, since these gave reasonable values for the saturation current. These are also within the uncertainty of an extrapolation from directly measured values but are \( \sim 0.1 \text{ eV} \) higher than those deduced from ion current measurements.\(^5\) This difference is well within the experimental irreproducibility of work-function measurements.

It is possible for the plasma to alter the electrode work functions.\(^4,54\) Because the ionization fraction is small for the cases considered herein, this effect should be negligible.

The agreement between theory and experiment in Fig. 7 is excellent. The curves shown cover \( pd = 34-135 \text{ mm Pa} \).
(10–40 mil Torr) which is typically the range of optimum performance for an ignited-mode thermionic energy converter. The agreement in current-voltage curves does not verify that the predicted plasma temperature and density profiles agree with experiment. Such proof requires experimental data on those variables.

A difference between this theory and experiment are expected for current very high or very low. At the highest current values shown in Fig. 7, the experimental curves are slightly above the theory, this may be due to the neglect of ion current. Further, at very low currents, the plasma density is low and surface ionization should be included. Because of this, the theoretical curves are not shown at currents below a couple of A/cm².

The purpose of an analytical model is as much to explain experiment as to agree with it. In particular, the relative importance of various causes of the arc drop can be explained using this model. First, consider Eq. (30) which relates the arc drop to heat transfer losses and collisional losses. These are plotted in Fig. 8, for the case of \( p = 270 \) Pa (2 Torr), \( d = 0.254 \) mm (10 mil), and other conditions the same as for Fig. 7. Curve 1 is the voltage loss due to heat transfer to the walls, \( (2k/e)(J_e/J)(T_e - T_0) \). \( J_e/J \) varies from \( -1.5 \) in the double-valued sheath region to \( -1.1 \) at high currents in the single-valued sheath region. \( T_e - T_0 \) varies from \( \sim 1000 \) K to \( \sim 1200 \) K at high currents. Consequently, this heat transfer term varies little. By contrast, the collisional energy loss term, \( Q/J \), varies greatly as current changes. At low currents, \( Q/J \) is less important than wall losses. At high currents, \( Q/J \) increases rapidly with current becoming the dominant loss mechanism. This is because high currents are possible only if the Schottky effect is significant and thus requires very high charged-particle densities. High charged-particle densities require a substantial power into ionization and hence a large \( Q \).

Also shown in Fig. 8 is \( -\Delta \phi \), the voltage “loss” due to the change in cathode work function from its zero field to its effective value. At low currents, this is negative showing that the double-valued sheath provides a voltage “gain.” At high currents, \( -\Delta \phi \) is positive because the Schottky effect lowers the work function. This voltage loss is small compared to the increase in current it causes.

Equation (35) provides an alternative view of the arc drop as being due to an ohmic loss and sheath/diffusion effects; these are plotted in Fig. 9. Curve 2 shows the ohmic loss \( J R \). At low currents, \( J R \) is proportional to \( J/n_{MAX} \) and is approximately constant. In the single-valued sheath region, the electron density rises faster than current in order to provide the Schottky emission necessary for larger currents. Consequently, \( J R \) declines at high currents. The sheath/diffusion loss, shown as curve 1, behaves differently. In the double-valued sheath region it is slightly smaller than the ohmic drop. In the single-valued sheath region, it rises rapidly with current becoming the dominant loss mechanism. This is expected since a large cathode sheath is necessary to draw large currents.

The net ionization rate per unit volume, \( S\bar{n} - \alpha n^3 \), is plotted against position in Fig. 10. Ionization is slow near the boundaries because \( \bar{n} \) is small there. At low currents, the net ionization rate peaks at the center, \( x = d/2 \). At higher cur-
currents, the rate peaks off center and has a local minimum at \( x = d/2 \). This is because recombination is significant at higher currents and peaks at the center. Ratafia and Keck\(^{56}\) introduced an analytical model of a thermionic discharge based on approximating the net ionization rate as a trigonometric function with three adjustable parameters. The form they chose has a single peak and, as anticipated by Ratafia and Keck\(^{56}\) is useful for small \( d \) or low currents for which recombination is small.

The cathode sheath behavior strongly affects the current-voltage curve. This is because the Schottky effect can significantly increase the emitted current, \( J_E \), while the double-valued sheath can reduce it. The double-valued sheath effect is controversial because of the possibility of trapped ions\(^{59}\) which might occur as follows. While the potential peak in the cathode sheath, as shown in Fig. 1, is a barrier to electrons, it is a potential well to ions. If ions become trapped in this well, their positive charge would reduce the height of the barrier. While ions entering the sheath from the plasma have too much energy to be trapped in the sheath, if an ion undergoes charge exchange with a low-energy atom while in the sheath, the resulting ion could be energetically trapped. The overall effect of these trapped ions is to reduce the potential peak and bring \( J_E \) closer in value to the Richardson current.

The results discussed so far have included both the Schottky and double-valued sheath effects. Comparison between the theory with and without these effects is shown in Fig. 11 for \( p = 270 \) Pa (2 Torr) and other conditions as for Fig. 7. At a current density of \( \sim 4.5 \) A/cm\(^2\), the emitted current is the Richardson current in both cases and the curves agree. Above this current, the inclusion of the Schottky effect results in a higher current and better agreement with experiment. Below this current, the theory with the double-valued sheath predicts that the output voltage \( V \) increases as the current \( J \) declines. This is because (1) the double-valued sheath increases the effective work function causing a larger output voltage, and (2) the double-valued sheath reduces \( J_E \) which reduces the plasma arc drop [see Eq. (30)]. Without the double-valued sheath, the work...
The good agreement between the theory including the double-valued sheath and experiment may be misleading. Suppose there was only a single-valued sheath even at low currents. The lower current branch of the single-valued sheath current-voltage curve, as shown in Fig. 11, is unstable for a normal load. Consequently, attempts to reduce the current below the stable branch may result in a constricted discharge with part of the discharge operating in the ignited mode and part in the diffusion mode. As the current is reduced, the area occupied by the ignited mode declines until a complete transition to the diffusion mode is made. The resulting current-voltage curve may also be similar to experiment. Further research is needed to decide this question.

VI. CONCLUSIONS

The isothermal assumption has proved useful for analyzing the plasma of a thermionic discharge. The model found reduced to three algebraic relations each stating a physical principle. The first, Eq. (26), determines the electron temperature necessary for ionization to balance diffusion to the walls and recombination. The second, Eq. (30), determines the arc drop necessary for electrical heating of the electrons to balance heat transfer to the walls and heat loss by collision with heavy particles. The third, Eq. (37), independently determines the electron temperature from heat dissipation in electrical resistance, sheath drops, etc., which depend on the electron density. For a given current, these three equations determine the electron temperature, the plasma arc drop, and the peak electron density. From these, the electron density distribution can be found from Eq. (23). Excellent agreement with experimental current-voltage characteristics was found.

The model permitted the causes of the arc drop to be identified. In the double-valued sheath regime, most of the arc drop was due to heat transfer between the hot electrons and the cooler electrodes. For high currents, most of the arc drop was due to the power needed for ionization. The high ionization rates are necessary because a high electron density is needed to produce a significant Schottky effect which permits high emitted currents.

At present, the model has no internal check on the accuracy of the isothermal assumption. The work of Rason provides a first step in this direction.

The discussion of the current-voltage characteristics with a single-valued sheath assumed indicated possible discharge constriction. Further research is required to clarify when the low current behavior is due to a double-valued sheath and when there are trapped ions and constriction occurs.

In addition to the quantitative model developed in Sec. III, a "plasma ideal" thermionic discharge was defined in Sec. IV and its current-voltage characteristics were compared with the well-known vacuum-ideal thermionic diode.

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APPENDIX: PLASMA BOUNDARY CONDITIONS

The density boundary conditions of Eq. (12) are sufficiently accurate except when performing the integral in Eq. (35) to find the plasma resistance. The more accurate boundary conditions needed to perform this integral are discussed in this appendix. At the boundary between the quasineutral plasma and the sheaths, the particle densities and fluxes must be continuous. This forms the physical basis for the considerations that follow.

The boundary conditions depend on the plasma sheath. It will be assumed that (a) the Debye length is much smaller than a mean free path, (b) each sheath is electron retaining, (c) the sheath height is large compared to the average electron thermal energy, and (d) the electrode is approximately planar. Matching the fluxes $\Gamma_e$ and $\Gamma_i$ at $x = 0$ + in the quasineutral plasma with the electron and ion fluxes in the cathode sheath yields:

$$\Gamma_e = \Gamma_E \left[ n(0+) \bar{v}_e / 4 \right] \exp(-eV_e/kT_e), \quad (45)$$
$$\Gamma_i = -\alpha_k n(0+) \bar{v}_e / 3.$$

Similarly, matching particle fluxes into the anode sheath at $x = d$ -:

$$\Gamma_e = \left[ n(d-) \bar{v}_e / 4 \right] \exp(-eV_A/kT_e), \quad (47)$$
$$\Gamma_i = \alpha_A n(d-) \bar{v}_i / 4,$$

where $\Gamma_E$ is the electron flux emitted thermionically from the cathode, $V_e$ and $V_A$ are the cathode and anode sheath heights, respectively, $n(0+)$ and $n(d-)$ are the quasineutral plasma densities at $x = 0$ + and $x = d$ - respectively, $\bar{v}_e$ and $\bar{v}_i$ are the electron and ion mean thermal speeds at the edge of the quasineutral plasma. If the sheath is single-valued, $\Gamma_e$ is given by the Richardson equation possibly as modified by the Schottky effect. Evaluation of $\Gamma_E$ in the double-valued sheath region requires consideration of the space-charge distribution in the sheath. The ion currents to each electrode are larger in magnitude than their thermal fluxes by factors $\alpha_e$ and $\alpha_k$ which can be found from Bohm's criterion. Equations (45) and (47) can be solved for the plasma sheath drops, as in Eq. (32), which were used to find the arc drop in Eq. (34).

The boundary conditions for the diffusion equation can be found as follows. From transport equations, Eq. (1) and Eq. (2), it is found that

$$-D_e \frac{dn}{dx} = \Gamma_i + \mu_i \Gamma_e / \mu_e.$$  \hspace{1cm} (49)

Combining Eq. (49) with Eqs. (46) and (48) and rearranging, the densities at $x = 0$ + and $x = d$ - can be found in terms of the density gradient at these locations:

$$n(0+) = \left. \frac{4}{\alpha_k \bar{v}_e} \frac{D_e}{dx} \right|_{x=0+} - \frac{\mu_i}{\mu_e} \Gamma_e,$$
$$n(d-) = \left. \frac{4}{\alpha_A \bar{v}_i} \frac{D_e}{dx} \right|_{x=d-} - \frac{\mu_i}{\mu_e} \Gamma_e.$$  \hspace{1cm} (50)

By dimensional analysis, it can be shown that the above boundary conditions imply that the electron density at the edges compared to typical densities in the bulk in $O(\lambda/d)$. This verifies that Eqs. (12) are a good first approximation.

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[References are omitted for brevity. The full text contains references to the works cited.]